RedoC's Math Note
Week 1 - Analysis

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Abstract

Welcome! This is the first week of RedoC's Math Note. Because this note would be your first meeting with analysis, I'm going to tell you why we should learn analysis in this note. Are you ready? Here we go!

1. What is Analysis?

What is analysis? This question is one of the simplest but powerful questions to dive into a new field. Well, to answer this question, we should know the outline of the history of analysis. Starting with ancient Greece, Zeno of Elea came up with a famous paradox, *Zeno's paradox of the dichotomy*. Later, the method of exhaustion was invented by Greek mathematicians to calculate the area and volume of regions and solids. Modern mathematical analysis started in 17th-century Europe. Fermat and Descartes developed analytic geometry, which allowed for analysis to be established. In the 18th century, Euler introduced the notion of a mathematical function, and Cauchy used infinitesimals to formulate calculus. From the contributions of these mathematicians, the (ϵ, δ)-definition approach was introduced. At the same time, Riemann came up with integration theory. Since then, many mathematicians have dealt with concepts of continuity, limits, differentiation, integrations, and more. In the early 20th century, Lebesgue made a significant advance in measure theory, and Banach created functional analysis.[1] As we can see in the history of analysis, analysis is the branch of mathematics dealing with continuity using concepts from calculus.

Keywords: Analysis, (ϵ, δ)-definition, Riemann series theorem
Special Thanks to my KSA friends: '공부하는 사람들'.

[1]It is summary of the wikipedia page [1]. I feel like being a copycat :P

2. **Can you explain?**

To tell you the importance of analysis, I brought up a fun problem. Many of you might know the math YouTuber 'Logical 로지컬.' One of his main contents is making videos that seem wrong intuitively but are hard to refute. Among those videos, I want to introduce the video 'Video that proves 1+1=1', the most viewed video in the channel.

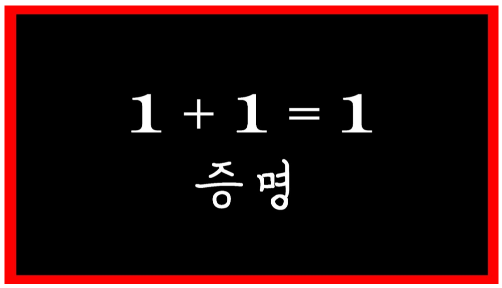
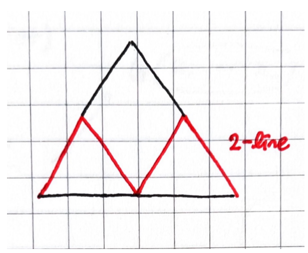


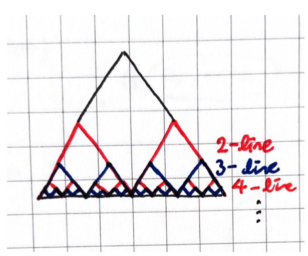
Figure 1. Thumbnail of 'Video that proves 1+1=1'

Let me give you a summary of proof in the video.

Proof. Let's think of an equilateral triangle in $\mathbb{R}^2$ whose every side length is 1. Then, we can put the connected line with two upper adjacent sides as 1-*line*. Of course, its length is 2. What happens if we connect three midpoints as follows and derive 2-*line*?

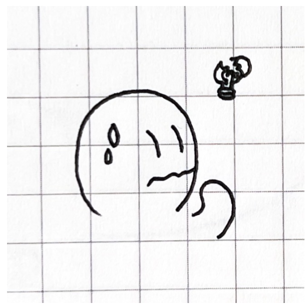


Though the length of 2-*line* is also 2, 2-*line* must be closer to the base than 1-*line*. Let's repeat to connect midpoints of the triangles made by $n-1$-*line* and make n-*line*.



Using basic geometric knowledge, we can get the length of every n-*line* is 2. However, because n-*line* converges to the base whose length is 1 as n grows up, the length of n-*line* is 1, too. Therefore, $1 = 2$. $\square$

It is obvious that $1 + 1$ is 2, not 1. It means that the above video must have some errors. *Can you explain why the proof is wrong?*



Hmmm...

Let's mathematically rewrite the problem. Let $x(n)$ and $L(x)$ be the n-*line* and the length of x, respectively. The proof is showing that for every n, $L(x(n)) = 2$. Using this fact, it claims $\lim_{n\to\infty} L(x(n)) = 2$.

However, there is a problem. We should get **the length of n-*line* when** $n \to \infty$, $L(\lim_{n\to\infty} x(n))$, not the limit of $L(x(n))$ when $n \to \infty$, $\lim_{n\to\infty} L(x(n))$. Moreover, we can't sure $L(\lim_{n\to\infty} x(n)) = \lim_{n\to\infty} L(x(n))$. Actually, we can show $L(\lim_{n\to\infty} x(n)) \neq \lim_{n\to\infty} L(x(n))$.

Let's put x_∞ as the triangle base. Let's recall (ϵ, δ)-definition we learned in KSA. Let $a(n)$ and L be curves[2] in $\mathbb{R}^2$, then

$$\lim_{n\to\infty} a(n) = L \Leftrightarrow \forall\epsilon > 0\ \exists M > 0\ s.t.\ \forall x > M\ F(a(x), L) < \epsilon$$

when $F(X, Y)$ is the Fréchet distance[3] between X and Y. Now, let's show $\lim_{n\to\infty} x(n) = x_\infty$. For an arbitrary $\epsilon > 0$, $M > \log_2 \frac{\sqrt{3}}{\epsilon}$ satisfies (ϵ, δ)-definition[4]. Therefore $L(\lim_{n\to\infty} x(n)) = 1$, which implies $L(\lim_{n\to\infty} x(n)) \neq \lim_{n\to\infty} L(x(n))$.

This problem came from a common mistake to think $\lim_x g(f(x)) = g(\lim_x f(x))$. Studying analysis can prevent making hard-to-find mistakes like this.

[2]A curve is a continuous map from the unit interval into $\mathbb{R}$

[3]Check out Fréchet distance for extra explanation

[4]Check out Section 5 for extra explanation

3. Common Mistakes about Infinite Series

Wasn't Section 2 too easy? For nerds who want more, here is another common mistake in dealing with infinite series. Let's think of the finite sum of a finite sequence $(a_n)_{n=1}^{m}$:

$$\sum_{n=1}^{m} a_n = a_1 + a_2 + ... + a_m. \tag{3.1}$$

We know the result of summation doesn't change if the order of the series changes. Mathematically, for every automorphism ϕ of $\{1, 2, ..., m\}$,

$$\sum_{n=1}^{m} a_n = \sum_{n=1}^{m} a_{\phi(n)}. \tag{3.2}$$

stands. Will it be valid if we extend the range to infinite? Many people expect it will also work in the infinite world. Let's see if it works.

Let's think an infinite sequence $(b_n)_{n=1}^{\infty}$ defined by $b_n = (-1)^{(n-1)}\frac{1}{n+2}$. Then the series of $(b_n)_{n=1}^{\infty}$ is

$$\sum_{n=1}^{\infty} b_n = \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + ... \tag{3.3}$$

Because (3.3) converges, we can get the actual summation of it[5], but let's just make sure the summation is positive. If we think of $\phi \in Aut(\mathbb{N})$ such that

$$\phi(n) = \begin{cases} \frac{3n-1}{2} & 2 \nmid n \\ \frac{3n}{4} & n \equiv 0 \bmod 4 \\ \frac{3n+2}{4} & n \equiv 2 \bmod 4 \end{cases}, \tag{3.4}$$

we can change the order like

$$\sum_{n=1}^{\infty} b_{\phi(n)} = \frac{1}{3} - \frac{1}{4} - \frac{1}{6} + \frac{1}{5} - \frac{1}{8} - \frac{1}{10} + \frac{1}{7} + ... \tag{3.5}$$

Unexpectedly, this series converges to the negative value[6]. In fact, we can make this series converge to an arbitrary number by rearranging it according to the *Riemann series theorem* [2]! It shows that *changing the order of infinite series is not allowed*. Later, we'll learn what kinds of infinite series are allowed to change the order and prove the *Riemann series theorem*.

[5]The actual value is $\ln 2 - \frac{1}{2}$

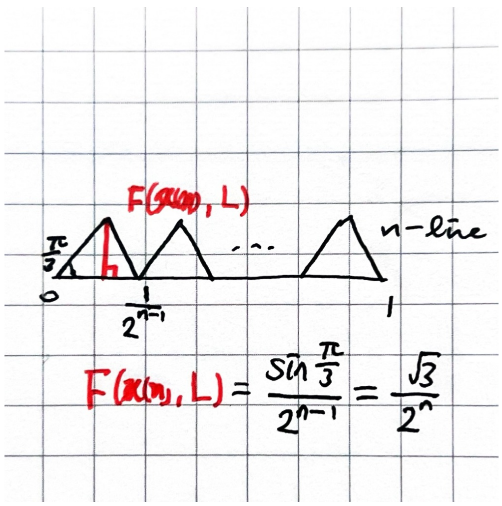
[6]The actual value is $\frac{\ln 2 - 1}{2}$

4. And More...

Analysis has so many branches such as real analysis, complex analysis, harmonic analysis, and functional analysis. And analysis is being applied to many fields in mathematics like number theory, combinatorics, topology, and geometry (I'm willing to introduce these applied fields!). If you major in mathematics, it will be inevitable to study analysis. I hope this note to be a perfect bridge between you and analysis.

5. Extra Explanation for Section 2

This is the extra explanation lots of students asked me to add. Let's put $a(x) = x(n)$ and $L = x_\infty$ from the (ϵ, δ)-definition. The following image indicates $F(x(n), L) = \frac{\sqrt{3}}{2^n}$.



For $\frac{\sqrt{3}}{2^n}$ to be less than ϵ where $n > M$, M must be greater than $\log_2 \frac{\sqrt{3}}{\epsilon}$. Thus, for an arbitrary $\epsilon > 0$, $M > \log_2 \frac{\sqrt{3}}{\epsilon}$ satisfies (ϵ, δ)-definition.

References

[1] *Mathematical analysis.* Wikipedia, January 2024.
[2] T. Tao, *Tao* 해석학, vol. Ⅰ. Hanbit Academy, Inc., 2023.

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