RedoC's Math Note
Week 2 - Differential Geometry

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Abstract

Welcome everyone! This is the second week of RedoC's Math Note. In this note, we learn what *Differential Form* is. After reading this, you'll be able to answer what dx and dy mean.

1. Introduction

First, I want to announce that RedoC's Math Note won't deal with only Analysis. As I study various branches of mathematics, there are so many things I want to introduce to you. So I concluded restricting topics of RedoC's Math Note to Analysis doesn't help you and me. Please understand this decision. But most of you wanted to learn Analysis, so I would try my best to choose topics in Analysis.

Before starting the note, I want to tell you my story. When I studied Calculus or Calculus-adapted Physics, I was skeptical to think of $\frac{dx}{dy}$ as a fraction dx divided by dy. Also, I doubted the calculations in Physics using the differentials, like dl, dA, and dV. Of course, they made sense intuitively, but I wanted *strictness*.

I'm not sure if anyone has similar experience, but it would be better to think about the following questions.

- What is dx?
- What does $\frac{dy}{dx}$ mean?
- Is it okay to change $f'(x) = \frac{df}{dx}$ into $df = f'(x)dx$?

And Differential Form will help you to answer these questions.

Keywords: Differential Geometry, Differential Forms

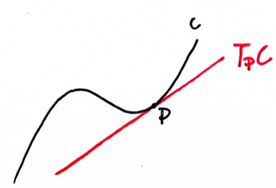
Special Thanks to my KSA friends: '공부하는 사람들'.

2. **Tangent Space and One-form**

We start our discussion with the definition of tangent space.

Definition 2.1. Suppose $C \subseteq \mathbb{R}^n$ is a curve and $P \in C$. The tangent space of C at P, $T_P C$ is the set of all vectors tangent to C at P.

For instance, the tangent space is a line if $n = 2$ like the following image.

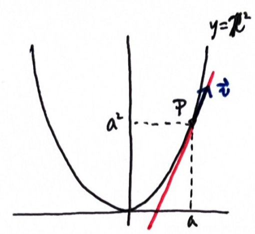


If we let the tangent vector whose x component is 1 be p, we can easily figure out $\vec{p} = \langle 1, f'(a) \rangle$. Because $\vec{p}$ is the basis of $T_P C$, $T_P C = span\{\vec{p}\} = \{\langle c, cf'(a)\rangle : c \in \mathbb{R}\}$. Now we define the coordinate system on $T_P C$.

Definition 2.2 ($n = 2$). Let $dx, dy : T_P C \to \mathbb{R}$ be the coordinate functions of $T_P C$. Then, we define the coordinate system on $T_P C$ as

$$\langle dx, dy\rangle : T_P C \to \mathbb{R}^2 \quad \langle dx, dy\rangle (v) = \langle dx(v), dy(v)\rangle \quad (v \in T_P C).$$

Let's look at an intuitive example. Let C be a parabola which is represented as $y = x^2$ algebraically. The following image indicates C, $P\left(a, a^2\right) \in C$ and $\vec{v}$.



Because the slope of C at P is $2a$, $\vec{v} = \langle c, 2ac\rangle$. By Definition 2.2, $dx(v) = c$ and $dy(v) = 2ac$. In short, $dy = 2a \cdot dx$. Thus, $\frac{dy}{dx} = 2a = f'(a)$.

Let's think about $T_P\mathbb{R}^2$. Because every vector is the tangent vector of the point in $\mathbb{R}^2$,

$$T_P\mathbb{R}^2 = span\left\{\langle 1,0\rangle, \langle 0,1\rangle\right\} = \left\{\langle dx, dy\rangle : dx, dy \in \mathbb{R}\right\}.$$

Now we define the one-form concept.

Definition 2.3. A one-form is a linear function $\omega : T_P\mathbb{R}^n \to \mathbb{R}$

Because every ω must be linear, if $n = 2$,

$$\begin{aligned}\omega\left(\langle dx, dy\rangle\right) &= adx + bdy = \langle a, b\rangle \cdot \langle dx, dy\rangle \\ &= \|\langle a, b\rangle\|\, comp_{\langle a,b\rangle} \langle dx, dy\rangle\end{aligned} \tag{2.1}$$

holds. We can get an intuition here: *A one-form is a multiple of the scalar projection onto some line.* Let's generalize (2.1) and rewrite Definition 2.3.

Definition 2.4. A one-form is a linear function $\omega : T_P\mathbb{R}^n \to \mathbb{R}$. Because of its linearity,

$$\omega = \sum_{i\in I\subseteq(1,2,\ldots,n)} a_i dx_i \quad (a_i \in \mathbb{R}) \tag{2.2}$$

stands.

3. **Differential Form**

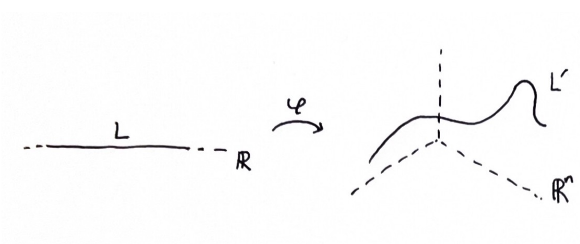
Let's define the differential one-form concept from Definition 2.4.

Definition 3.1. A differential one-form on $\mathbb{R}^n$, ω is defined by following equation:

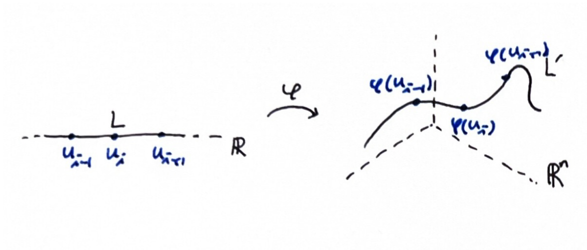
$$\omega = \sum_{i\in I\subseteq(1,2,\ldots,n)} f_i dx_i, \tag{3.1}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$ is differentiable.

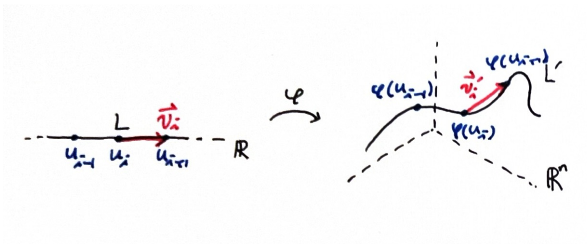
Don't be afraid of the definition. $f(x)dx$ is a kind of differential form. So, how can we use this differential form? Let's define a smooth map $\varphi : L \to L'$. $L \subseteq \mathbb{R}$ parameterizes $L' \subseteq \mathbb{R}^n$ via φ. The following image indicates it.



How can we approximate tangent vectors of L'? Let's make lattices on L. Each point of lattices on L is located at (u_i). Because φ wraps L into L', the lattice on L' corresponding with (u_i) locates at $\varphi(u_i)$.



Now, let's think a vector $\vec{v_i} = (u_{i+1}) - (u_i)$ and correspoinding vector $\vec{v_i}' = \varphi(u_{i+1}) - \varphi(u_i)$.



What if $\Delta u (:= u_{i+1} - u_i) \to 0$? Like our expectation, $\vec{v_i}'$ becomes *almost* a tangent vector[1], therefore we can think $\vec{v_i}' \in T_{\varphi(u_i)}\mathbb{R}^n$. Using these observations, let's derive a proposition.

PROPOSITION 3.1. *Let ω be a differential one-form. Then,*

$$\int_{L'} \omega = \int_L \omega_{\varphi(u)} \left(\frac{d\varphi}{du} \right) du \tag{3.2}$$

holds.

Proof.

$$\begin{aligned}
\int_{L'} \omega &= \lim_{\Delta u \to 0} \sum_i \omega_{\varphi(u_i)}(\varphi(u_{i+1}) - \varphi(u_i)) \\
&= \lim_{\Delta u \to 0} \sum_i \omega_{\varphi(u_i)} \left(\frac{\varphi(u_{i+1}) - \varphi(u_i)}{\Delta u} \right) \Delta u \\
&= \lim_{\Delta u \to 0} \sum_i \omega_{\varphi(u_i)} \left(\frac{d\varphi}{du} \right) \Delta u \\
&= \int_L \omega_{\varphi(u)} \left(\frac{d\varphi}{du} \right) du
\end{aligned}$$

□

[1] It seems obviously true, but it can be false if φ is not smooth.

There is a great corollary we learn in Calculus I.

Corollary 3.2 (u-substitution). *Let* $\varphi : [a, b] - > [\varphi(a), \varphi(b)]$, *then*

$$\int_{\varphi(a)}^{\varphi(b)} f(x)dx = \int_a^b f(\varphi(u))\varphi'(u)du. \tag{3.3}$$

Proof. Let $\omega = f(x)dx$ be a differential form. Then,

$$\int_{[\varphi(a),\varphi(b)]} \omega = \int_{\varphi(a)}^{\varphi(b)} f(x)dx \tag{3.4}$$

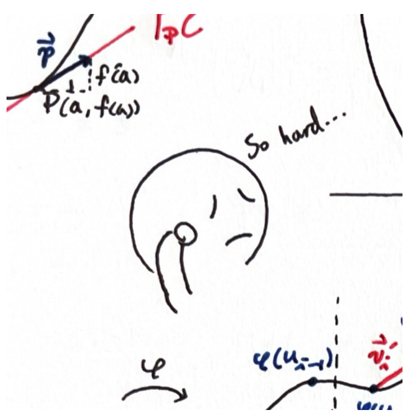
holds. However, if we use φ-parameterization,

$$\begin{aligned} \int_{[\varphi(a),\varphi(b)]} \omega &= \int_{[a,b]} \omega_{\varphi(u)}(\varphi'(u))du \\ &= \int_a^b f(\varphi(u))dx(\varphi'(u))du \\ &= \int_a^b f(\varphi(u))\varphi'(u)du \end{aligned} \tag{3.5}$$

holds. By (3.4) and (3.5), we get

$$\int_{\varphi(a)}^{\varphi(b)} f(x)dx = \int_a^b f(\varphi(u))\varphi'(u)du. \tag{3.6}$$

□



So hard...

4. **Exterior Derivative**

To connect a function[2] and a differential one-form, let's define a derivative d.

Definition 4.1. Let d be a mapping from differential zero-form to differential one-form. In other words, given a zero-form on $\mathbb{R}^n$, $f(x_1, x_2, ..., x_n)$, df is the differential one-form corresponding to f. We call df the exterior derivative of f.

Then, how can we evaluate such df? Recall every differential one-form needs $P \in \mathbb{R}^n$ and $\vec{v} \in T_P\mathbb{R}^n$ to be evaluated. Looking at these requirements, we can figure out the directional derivative will be perfect to evaluate df.

PROPOSITION 4.1 (evaluation of df).

$$df_{P(\vec{v})} = D_{\vec{v}}f|_{\vec{X}=\vec{P}} = \sum_{i\in(1,2,\ldots,n)} \frac{\partial f}{\partial x_i} v_i = \sum_{i\in(1,2,\ldots,n)} \frac{\partial f}{\partial x_i} dx_i(\vec{v}) \tag{4.1}$$

Let me give you a simple example. Think about a parabolic function $f(x) = x^2$. Then its exterior derivative is

$$df = \frac{\partial f}{\partial x} dx = \frac{df}{dx} dx = 2x dx \tag{4.2}$$

If we put $y(x) := f(x)$, $dy = 2xdx$ holds.

5. **Answer the questions**

Now, let's answer the questions in Section 1.

- What is dx? dx **is a coordinate function of** T_PC**.**
- What does $\frac{dy}{dx}$ mean? **The slope of the tangent vectors in** T_PC**.**
- Is it okay to change $f'(x) = \frac{df}{dx}$ into $df = f'(x)dx$? **Yes. We can get this as generalizing (4.2).**

[2] We assume a function to be a zero-form.

6. **And more...**

I tried not to use any difficult notions like tensor and manifold when writing this note. So there are lots of *logical* holes. Remember that we only learned a few of the fundamental concepts of Differential Form. For example, we can generalize the one-form concept into the m-form concept using wedge products and multiple integrals. After these prerequisites are ready, I'll introduce deeper concepts in differential geometry—Special thanks to Michael Penn[3] for providing great materials to study with.

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