RedoC's Math Note
Week 3 - Group Theory

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Abstract

Welcome everyone! This is the thrid week of RedoC's Math Note. It is unbelievable that this note will be the last note before entrancing into KSA. Woohoo! Because of this, I want this note to be helpful for studying in KSA. In this note, we learn what *Group* is. Sorry for making this note shorter than others due to the lack of time.

Keywords: Group Theory, Symmetric Group, Cayley's Theorem

Special Thanks to my KSA friends: '공부하는 사람들'.

1. Symmetric Group

Let's look at a definition of group.

Definition 1.1. A *group* is a collection of symmetries of something.

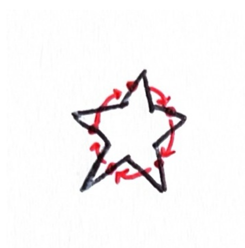
What is the definition of symmetry in Definition 1.1?

Definition 1.2. A *symmetry* is a map from something to itself *preserving* its structure.

We call groups satisfying Defintion 1.1 as symmetric groups.

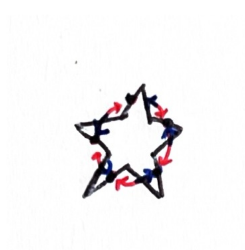
Definition 1.3. A *symmetric group* is a set consisted of all symmetries of something and group operation which is the composition of maps.

Shell we think about a star.



There are 5 symmetries of rotation. For this case, we say this group has order of 5. After you get the mentoring for group theory, you can also say this group is cyclic, but it is beyond our discussion.

There is the identity element(symmetry) in every group. For the above case, the identity element is just *staying still*. Quite trivial, isn't it? Every symmetry has its inverse. Look at the below figure.



Every blue arrow is the inverse element corresponding to its adjacent red arrow. We denote the symmetric group over a set X as $\mathrm{Sym}(X)$.

2. Group Axioms and the First Isomorphism Theorem

Now we look at another definition of group.

Definition 2.1. A group is nonempty set G together with the law of composition[1] $\circ$ satisfying these three requirements:

- $\forall a, b, c \in G, \quad (a \circ b) \circ c = a \circ (b \circ c)$ (associativity)
- $\exists e \in G \quad s.t. \quad \forall a \in G \quad e \circ a = a \circ e$ (identity element)
- $\forall a \in G \quad \exists a^{-1} \in G \quad s.t. \quad a^{-1} \circ a = a \circ a^{-1} = e$ (inverse element)

Looking at this, we can find out that these axioms are related to some properties of symmetries. However, there is a clear difference; symmetries are in the real world, but axioms are in the abstract world. Well, you must know that abstract algebra is more tolerant of equality. We need to know *isomorphism* to figure out what is *the equality in abstract algebra.*

Definition 2.2. Let G and G' be groups. A isomorphism φ is a *structure-preserving* bijective mapping $G \to G'$. Citing a great commutative diagram in [2],

$$\begin{tikzcd} (x, y) \arrow[r, "\circ"] \arrow[d, "\varphi\times\varphi"] & x \circ y \arrow[d, "\varphi"] \\ (\varphi(x), \varphi(y)) \arrow[r, "\circ"] & \varphi(x \circ y) = \varphi(x) \circ \varphi(y) \end{tikzcd} \tag{2.1}$$

holds. Intuitively,

$$\text{operation} \to \text{relabel} = \text{relabel} \to \text{operation}$$

holds. If there is a isomorphism between G and G', we say G and G' are isomorphic, denoted by $G \cong G'$.

We say given two groups are almost same if they are isomorphic. There is a very useful theorem about isomorphism, which is used to prove Cayley's theorem. I introduce some extra definitions and then give you the theorem.

Definition 2.3. A *normal subgroup* of G is a subgroup that is invarient under conjugation by members of G.

Definition 2.4. Let N be a normal subgroup of G. Define G/N to be the set of all left cosets of N in G. *id est*, $G/N := \{aN : a \in G\}$.

THEOREM 2.1 (first isomorphism theorem). *Let G and H be groups, and let $f : G \to H$ be a homomorphism. Then* $\operatorname{Im} f \cong G/\ker f$.

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[1] You can think a law of compostion as same as a binary operation. This is just kind of Lang-ish[1] word :)

3. Cayley's Theorem

Now it is showtime! Using the first isomorphism theorem (Theorem 2.1), we can prove that *Definition* 1.1 *is almost same with Definition* 2.1.

THEOREM 3.1 (Cayley). *Every group G is isomorphic to a subgroup of a symmetric group.*

Proof. Let's think of a group G as acting on itself by left multiplication, i.e. $gx = g \circ x$. The group action[2] of G must have its permutation representation, so we put the action as $\pi : G \to Sym(G)$. If π is injective, the theorem holds. Suppose $g \in \ker \pi$, then $g = ge = g \circ e = e$. Therefore $\ker \pi$ is trivial and it shows π is injective. By the first isomorphism theorem, following stands:

$$Sym(G) \supset \operatorname{Im} \pi \cong G/\ker \pi = G, \tag{3.1}$$

which is what we want to claim. □

There is easier proof for Cayley's theorem, but I think this proof is more condensed and intuitive. If you want to check out, visit the Wikipedia page [3] for additional explanation.

4. And more...

Because my favorite field is algebra, I had a hard time to choose the topic from so many interesting concepts and theorems. After a long thought, I've decided to give you another intuitive perspective to look at the group concept. So, I chose to introduce the relationship between the axiomatic defintion and the symmetry group concept. I hope you enjoyed reading this note :)

References

[1] S. Lang, *Algebra*. Springer, 2002.
[2] *대수학*, vol. II of *학부 대수학 강의*. 서울대학교출판문화원, 2022.
[3] *Cayley's theorem*. Wikipedia, November 2023.

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[2]For example, rotations around a point in the plane are examples of group action.